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Reflexivity defect of kernels of the elementary operators of length 2

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ABSTRACT

Let $\mathcal{X}$ be a finite-dimensional complex vector space. We give an explicit formula for the reflexivity defect of the kernel of an arbitrary elementary operator of length 2, i.e., an elementary operator of the form $\Delta(T) = A_1TB_1 - A_2TB_2$ $(T \in L(\mathcal{X}))$ where $A_1,\ A_2$ and $B_1,\ B_2$ are linearly independent.

1. Introduction

Let $\mathcal{X}$ be a finite-dimensional complex vector space and $\mathcal{S}$ be a non-empty subset of $L(\mathcal{X})$, the space of all linear transformations on $\mathcal{X}$. Let k be a positive integer. Define the k-reflexive cover of $\mathcal{S}$ to be the space

$$\mathrm{Ref}_k\mathcal{S} = \{T \in L(\mathcal{X}) : \forall\, \varepsilon > 0,\ \forall\, x_1, \ldots, x_k \in \mathcal{X} : \exists S \in \mathcal{S} :\ \|Tx_i - Sx_i\| < \varepsilon,\ \ i = 1, \ldots, k\}. \tag{1}$$

One can see that $\mathrm{Ref}_k\mathcal{S}$ is a linear subspace of $L(\mathcal{X})$. A linear subspace $\mathcal{S}$ is said to be k-reflexive if $\mathrm{Ref}_k\mathcal{S} = \mathcal{S}$. The k-reflexivity defect is defined by $\mathrm{rd}_k(\mathcal{S}) = \dim(\mathrm{Ref}_k\mathcal{S}) - \dim(\mathcal{S})$.

For a linear transformation $S \in L(\mathcal{X})$ let $S^\top$ denote the transpose of S. Let $A,\ B \in L(\mathcal{X})$ be invertible linear transformations and let $\mathcal{S}$ be a non-empty subset of $L(\mathcal{X})$. Let us denote $A\mathcal{S}B = \{ASB :\ S \in \mathcal{S}\}$ and $\mathcal{S}^\top = \{S^\top\ :\ S \in \mathcal{S}\}$. Let $\mathcal{S} \subseteq L(\mathcal{X})$ be a linear subspace. It is well known that transformations of the type

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$$\mathcal{S} \mapsto A\mathcal{S}B \quad \text{and} \quad \mathcal{S} \mapsto \mathcal{S}^{\mathsf{T}} \tag{2}$$

preserve the k-reflexivity defect, a fact we will use often.

Because the k-reflexivity defect is preserved by similarity transformations and $\mathcal{X}$ is a finite-dimensional complex vector space, one can assume that $\mathcal{X} = \mathbb{C}^n$ for some $n \in \mathbb{N}$. Thus, $L(\mathcal{X})$ may be identified with $\mathbb{M}_n$, the algebra of all n-by-n complex matrices. Throughout this paper we will be dealing with subspaces of $\mathbb{M}_n$ which are of the form

$$\mathcal{S} = \begin{pmatrix} \mathcal{S}_{11} & \dots & \mathcal{S}_{1N} \\ \vdots & & \vdots \\ \mathcal{S}_{M1} & \dots & \mathcal{S}_{MN} \end{pmatrix},$$

where, for each pair of indices (i, j), $\mathcal{S}_{ij}$ is a subspace of $\mathbb{M}_{m_i,n_j}$, the space of all m_i-by-n_j complex matrices, and $\sum_{i=1}^{M} m_i = \sum_{j=1}^{N} n_j = n$. It is not hard to see that for spaces of this type one has

$$\mathrm{Ref_k}(\mathcal{S}) = \begin{pmatrix} \mathrm{Ref_k}(\mathcal{S}_{11}) & \dots & \mathrm{Ref_k}(\mathcal{S}_{1N}) \\ \vdots & & \vdots \\ \mathrm{Ref_k}(\mathcal{S}_{M1}) & \dots & \mathrm{Ref_k}(\mathcal{S}_{MN}) \end{pmatrix} \quad \text{and} \quad \mathrm{rd}_k\,(\mathcal{S}) = \sum_{i=1}^{M}\sum_{j=1}^{N} \mathrm{rd}_k\,(\mathcal{S}_{ij})\,. \tag{3}$$

In particular, $\mathcal{S}$ is k-reflexive if and only if $\mathcal{S}_{ij}$ is k-reflexive for every pair of indices $i \in \{1, \dots, M\}$, $j \in \{1, \dots, N\}$.

In Section 2 we give the explicit formulae for the reflexivity defect of the generalized derivation and of the elementary operator of the form $\Delta(T) = ATB - T$ where $A, B, T \in \mathbb{M}_n$. Moreover, we consider the elementary operators of length 2 under some additional assumptions on the coefficients such as commutativity, normality and positive definiteness. We give an example of the elementary operator of length 2 with the coefficients from the set $\{J_n(0), J_n(0)^{\mathsf{T}}, I_n\}$ where $J_n(0)$ denotes the nilpotent Jordan block of order n and I_n denotes the n-by-n identity matrix. In Section 3 the basics of the well-known theory of the Kronecker structure of a linear matrix pencil are given. Using the Kronecker canonical form of a singular matrix pencil we obtain the explicit formula for the reflexivity defect of an arbitrary elementary operator of length 2 which is the main result of this paper. In section 4 an application of the result mentioned is given.

2. Elementary operators of length 2

Let (A_1, A_2) and (B_1, B_2) be arbitrary pairs of n-by-n complex matrices. The elementary operator on $\mathbb{M}_n$ with coefficients (A_1, A_2) and (B_1, B_2) is defined by

$$\Delta(T) = A_1TB_1 - A_2TB_2, \quad T \in \mathbb{M}_n. \tag{4}$$

It is not hard to see that the kernel of Δ is a 2-reflexive subspace of $\mathbb{M}_n$, i.e., $\mathrm{rd}_k(\ker \Delta) = 0$ for all $k \geqslant 2$. Hence, it is reasonable to ask whether $\mathrm{rd}_1(\ker \Delta)$ can be determined. In what follows, we are interested in the 1-reflexivity defect of the kernel of elementary operators of length 2, i.e., linear transformations of the form $\Delta(T) = A_1TB_1 - A_2TB_2 \quad (T \in \mathbb{M}_n)$, where A_1, A_2 and B_1, B_2 are linearly independent. In other words, Δ cannot be written as a two-sided multiplication. To shorten the notation we will write $\mathrm{rd}(\ker \Delta)$ instead of $\mathrm{rd}_1(\ker \Delta)$ and we will refer to 1-reflexivity defect simply as the reflexivity defect. First we introduce some notation. For $k \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, let $J_k(\alpha)$ denote the Jordan block of size k, i.e.,

$$J_k(\alpha) = \begin{pmatrix} \alpha & 1 & & \\ & \ddots & \ddots & \\ & & \alpha & 1 \\ & & & \alpha \end{pmatrix} \in \mathbb{M}_k.$$

For a given $T \in \mathbb{M}_n$ let p_T and m_T denote the characteristic and minimal polynomial of T, respectively. Let $A,\ B \in \mathbb{M}_n$ be arbitrary matrices. Define the generalized derivation on $\mathbb{M}_n$ with coefficients A and B by $\delta\ (T) = AT - TB$, $T \in \mathbb{M}_n$. In [10] Šul'man has proved that if the coefficients A and B are normal, then the kernel of δ is reflexive. Moreover, a result of Zajac in [12], states that $\ker \delta$ is reflexive if and only if all roots of the greatest common divisor of the minimal polynomials m_A and m_B are simple.

Let $J_{p_1}(\lambda_1) \oplus \cdots \oplus J_{p_N}(\lambda_N)$ be the Jordan canonical form of A, where $\sum_{i=1}^N p_i = n$ and $\lambda_1, \ldots, \lambda_N$ are not necessarily distinct eigenvalues of A. Similarly, let $J_{r_1}(\mu_1) \oplus \cdots \oplus J_{r_M}(\mu_M)$ be the Jordan canonical form of B, where $\sum_{i=1}^M r_i = n$ and $\mu_1, \ldots, \mu_M$ are not necessarily distinct eigenvalues of B. Let $R(i,\ j)$ be a non-negative integer defined by

$$R(i,\ j) := \begin{cases} \frac{1}{2}\min\{p_i,\ r_j\}\left(\min\{p_i,\ r_j\} - 1\right) & \text{if } \lambda_i = \mu_j, \\ 0 & \text{otherwise.} \end{cases}$$

The following, a little more general result than the one in [12], which has been mentioned above, can be obtained, cf. [1] and the preprint *Zajac, Reflexivity of intertwining operators in finite dimensional spaces*.

Proposition 2.1. *With the above notation, the reflexivity defect of* $\ker \delta$ *can be expressed as*

$$\mathrm{rd}(\ker \delta) = \sum_{i=1}^{N}\sum_{j=1}^{M} R(i,\ j).$$

In particular, $\ker \delta$ *is a reflexive space if and only if all roots of the greatest common divisor of the minimal polynomials m_A and m_B are simple.*

Proof. Since similarity preserves the reflexivity defect we can assume that

$$\ker \delta = \left\{ \begin{pmatrix} T_{11} & \ldots & T_{1M} \\ \vdots & & \vdots \\ T_{N1} & \ldots & T_{NM} \end{pmatrix} :\quad T_{ij} \in \mathbb{M}_{p_i,r_j},\quad J_{p_i}(\lambda_i)T_{ij} = T_{ij}J_{r_j}(\mu_j) \right\}.$$

Define $\delta_{p_i,r_j}(T) = J_{p_i}(\lambda_i)T - TJ_{r_j}(\mu_j)$ for $T \in \mathbb{M}_{p_i,r_j}$, the space of all p_i-by-r_j matrices. By (3) it suffices to determine $\mathrm{rd}(\ker \delta_{p_i,r_j})$. If $\lambda_i \neq \mu_j$, then by [8, Theorem 4] we have that δ_{p_i,r_j} is bijective and hence $\ker \delta_{p_i,r_j}$ is reflexive as it is a trivial space. If $\lambda_i = \mu_j$, then by [1, Lemma 4.5, Lemma 4.6] one has $\mathrm{rd}(\ker \delta_{p_i,r_j}) = \frac{1}{2}\min\{p_i,\ r_j\}\left(\min\{p_i,\ r_j\} - 1\right)$. Hence, $\mathrm{rd}(\ker \delta_{p_i,r_j}) = R(i,j)$, which completes the proof. $\square$

Let $A,\ B \in \mathbb{M}_n$ be as before the Proposition 2.1 and let ϵ be the elementary operator on $\mathbb{M}_n$ defined by $\epsilon\ (T) = ATB - T$, $T \in \mathbb{M}_n$. Then the following holds.

Proposition 2.2. *If* $1 \notin \sigma\,(A)\,\sigma\,(B) = \{\lambda_i\,\mu_j\ :\ \lambda_i \in \sigma(A),\ \mu_j \in \sigma(B)\}$, *then* $\ker \epsilon$ *is reflexive. Otherwise*

$$\mathrm{rd}\,(\ker \epsilon) = \sum_{\lambda_i\,\mu_j = 1} \frac{1}{2}\min\{p_i,\ r_j\}\left(\min\{p_i,\ r_j\} - 1\right).$$

Proof. Since A is similar to $J_{p_1}(\lambda_1) \oplus \cdots \oplus J_{p_N}(\lambda_N)$ and B is similar to $J_{r_1}(\mu_1) \oplus \cdots \oplus J_{r_M}(\mu_M)$ the formula in (3) yields that $\mathrm{rd}(\ker \epsilon) = \sum_{i=1}^N \sum_{j=1}^M \mathrm{rd}(\ker \epsilon_{ij})$, where $\epsilon_{ij}(T) = J_{p_i}(\lambda_i)\,T\,J_{r_j}(\mu_j) - T$ is the elementary operator acting on the space of p_i-by-r_j matrices. If T satisfies the equation $J_{p_i}(\lambda_i)\,T\,J_{r_j}(\mu_j) = T$, then T is an eigenvector of the Kronecker product $J_{r_j}(\mu_j)^{\mathsf{T}} \otimes J_{p_i}(\lambda_i)$ at eigenvalue 1. Since $\sigma\left(J_{r_j}(\mu_j)^{\mathsf{T}} \otimes J_{p_i}(\lambda_i)\right) = \{\lambda_i\,\mu_j\}$ we can conclude that if $\lambda_i\mu_j \neq 1$, then $\ker \epsilon_{ij} = \{0\}$. Otherwise, if $\lambda_i\mu_j = 1$, then $J_{p_i}(\lambda_i)$ and $J_{r_j}(\mu_j)$ are invertible and hence $\ker \epsilon_{ij} = \ker \tilde{\epsilon}_{ij}$, where $\tilde{\epsilon}_{ij}$

is a generalized derivation of the form $\tilde{\epsilon}_{ij}(T) = J_{p_i}(\lambda_i)T - TJ_{r_j}(\mu_j)^{-1}$. Using [6, Example 6.2.13] it is easy to see that inverting matrices preserves sizes of Jordan blocks, hence the result follows by Proposition 2.1. □

Let $\Delta(T) = A_1TB_1 - A_2TB_2$ ($T \in \mathbb{M}_n$), where A_1, A_2 and B_1, B_2 are linearly independent. In what follows we consider reflexivity of $\ker\Delta$ under some additional assumptions on the coefficients. If A_1, A_2 commute and B_1, B_2 commute and $\sigma(A_1)\sigma(B_1) \cap \sigma(A_2)\sigma(B_2) = \emptyset$, then Δ is bijective by [8, Theorem 4] and hence $\ker\Delta$ is reflexive since it is a trivial subspace. Next, if $A_1, A_2, B_1, B_2 \in \mathbb{M}_n$ are normal and if A_1, A_2 commute and B_1, B_2 commute, then $\ker\Delta$ is reflexive. Indeed, this is an easy consequence of the fact that a commuting family of normal matrices is simultaneously unitarily diagonalizable [5, Theorem 2.5.5]. In the following proposition the commutativity assumption is replaced by some other condition.

Proposition 2.3. *Let $A_1, A_2, B_1, B_2 \in \mathbb{M}_n$ be hermitian matrices and let $\Delta(T) = A_1TB_1 - A_2TB_2$ ($T \in \mathbb{M}_n$). If at least one of A_1, A_2 and at least one of B_1, B_2 is positive definite, then* $\ker\Delta$ *is reflexive.*

Proof. If A_1, B_1 is a pair of positive definite matrices, then $\ker\Delta = \ker\tilde{\Delta}$ where $\tilde{\Delta}(T) = A_1^{-1}A_2TB_2B_1^{-1} - T$. By [4, Corollary 2.2] both matrix coefficients are diagonalizable and therefore by Proposition 2.2 $\ker\Delta$ is reflexive. Similarly, if A_1, B_2 is a pair of positive definite matrices, $\ker\Delta = \ker\tilde{\Delta}$ where $\tilde{\Delta}(T) = A_1^{-1}A_2T - TB_1B_2^{-1}$. Again, [4, Corollary 2.2] and Proposition 2.1 yield that $\ker\Delta$ is reflexive. The proof is similar in the remaining two cases. □

Next, we present a preliminary result which will prove itself useful later on. First, we introduce some notation. Let m, n be non-negative integers. Denote by $\mathcal{T}(m,n)$ the space of all $m \times n$ Toeplitz matrices and by $\mathcal{UT}(n)$ the algebra of all upper triangular n-by-n Toeplitz matrices. Let $m \leqslant n$ and introduce the following subspaces of $\mathbb{M}_{m,n}$

$$\mathcal{A}(m,n) = \left\{ \begin{pmatrix} a_1 & \cdots & \cdots & \cdots & a_n \\ & \ddots & & & \vdots \\ & & a_1 & \cdots & a_{n-m+1} \end{pmatrix} : a_i \in \mathbb{C} \right\},$$

$$\mathcal{B}(m,n) = \left\{ \begin{pmatrix} a_1 & \cdots & a_{n-m+1} & & \\ & \ddots & & \ddots & \\ & & a_1 & \cdots & a_{n-m+1} \end{pmatrix} : a_i \in \mathbb{C} \right\}.$$

Proposition 2.4. *Let m and n be positive integers.*

(i) *If $m \leqslant n$, then* $\mathrm{rd}(\mathcal{A}(m,n)) = (m-1)(n - \frac{1}{2}m)$ *and $\mathcal{B}(m,n)$ is a reflexive space,*
(ii) $\mathrm{rd}(\mathcal{UT}(n)) = \frac{1}{2}n(n-1)$,
(iii) $\mathrm{rd}(\mathcal{T}(m,n)) = (m-1)(n-1)$. *In particular, if $m = 1$ or $n = 1$, then the space $\mathcal{T}(m,n)$ is reflexive, otherwise* $\mathrm{Ref}\,\mathcal{T}(m,n) = \mathbb{M}_{m,n}$.

Proof. We first prove assertion (i). Let $m \leqslant n$ be positive integers. Note that $\mathrm{rd}(\mathcal{A}(m,n)) = \mathrm{rd}(\mathcal{A}^{\mathsf{T}}(m,n))$, where $\mathcal{A}^{\mathsf{T}}(m,n) = \{S^{\mathsf{T}} : S \in \mathcal{A}(m,n)\}$. Computing the reflexive cover of the latter space is equivalent to solving a system of linear equations with the property that the unknowns can be eliminated consecutively. Hence, one can easily see that

$$\mathrm{Ref}\,\mathcal{A}(m,n) = \left\{ \begin{pmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1n} \\ & \ddots & & & \vdots \\ & & a_{mm} & \cdots & a_{mn} \end{pmatrix} : a_{ij} \in \mathbb{C},\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n \right\}.$$

Since $\dim(\mathcal{A}(m,n)) = n$ and $\dim(\mathrm{Ref}(\mathcal{A}(m,n))) = m(n-\frac{1}{2}(m-1))$, one has $\mathrm{rd}(\mathcal{A}(m,n)) = (n-\frac{1}{2}m)(m-1)$. A result of Meshulam and Šemrl in [9] states that a d-dimensional space of operators is reflexive if all non-zero operators in it have rank at least $d+1$ and the underlying field is algebraically closed. Note that $\dim(\mathcal{B}(m,n)) = n-m+1$ and that every non-zero element of $\mathcal{B}(m,n)$ is of rank m. A brief calculation shows that if $n \leqslant 2(m-1)$, then $\mathcal{B}(m,n)$ is a reflexive space by [9, Theorem 1.1]. Assume therefore that $n > 2(m-1)$. The structure of the space $\mathcal{B}(m,n)$ yields that the elements of $\mathrm{Ref}\,\mathcal{B}(m,n)$ have zeros in the same positions as the elements of $\mathcal{B}(m,n)$, i.e.,

$$\mathrm{Ref}\,\mathcal{B}(m,n) = \left\{ \begin{pmatrix} * & \dots & * & & \\ & \ddots & & \ddots & \\ & & * & \dots & * \end{pmatrix} : \ * \text{ non-zero entry} \right\}.$$

Let $B \in \mathrm{Ref}\,\mathcal{B}(m,n)$ and by the discussion above we can assume that

$$B = \begin{pmatrix} b_{11} & \dots & b_{1,n-m+1} & & \\ & \ddots & & \ddots & \\ & & b_{mm} & \dots & b_{mn} \end{pmatrix}$$

for some $b_{ij} \in \mathbb{C}$. Let $t \in \mathbb{C}$ be a non-zero scalar and let $x = \begin{pmatrix} 1 & t & \dots & t^{n-1} \end{pmatrix}^{\mathsf{T}} \in \mathbb{M}_{n,1}$. By the definition of the space $\mathrm{Ref}\,\mathcal{B}(m,n)$ there exists an $S \in \mathcal{B}(m,n)$ of the form

$$S = \begin{pmatrix} a_1 & \dots & a_{n-m+1} & & \\ & \ddots & & \ddots & \\ & & a_1 & \dots & a_{n-m+1} \end{pmatrix}$$

such that $Bx = Sx$. This yields the following m equations

$$\sum_{j=0}^{n-m} b_{k,k+j}t^j = \sum_{j=0}^{n-m} a_{j+1}t^j \quad (k = 1, 2, \dots, m).$$

Subtracting the first equation from the rest gives

$$\sum_{j=0}^{n-m} (b_{k,k+j} - b_{1,1+j})t^j = 0 \quad (k = 2, \dots, m).$$

Since t is arbitrary we have $b_{k,k+j} = b_{1,1+j}$ for every $k = 2, \dots, m$ and every $j = 0, \dots, n-m$, hence $B \in \mathcal{B}(m,n)$. Note that this proof works not only for $n > 2(m-1)$ but for $n \geqslant m$, as well. The assertion (ii) is a well-known fact, however, it also follows from (i) since $\mathcal{A}(n,n) = \mathcal{UT}(n)$. The assertion (iii) is trivial to prove since reflexive cover respects the inclusion relations, i.e., if $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $\mathrm{Ref}\,\mathcal{S}_1 \subseteq \mathrm{Ref}\,\mathcal{S}_2$. $\square$

For $k \in \mathbb{N}$ let I_k denote the identity matrix of size k and let P_k denote the standard involutory permutation matrix of size k, i.e.,

$$I_k = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad P_k = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in \mathbb{M}_k.$$

Let us consider the following example.

Example 2.5. Let Δ be a linear transformation of the form $\Delta(T) = A_1TB_1 - A_2TB_2$, $(T \in \mathbb{M}_n)$, where $A_1, A_2, B_1, B_2 \in \{J_n(0), J_n(0)^{\mathsf{T}}, I_n\}$. We consider all the possible different forms of Δ where the coefficients A_1, A_2 and B_1, B_2 are linearly independent. That is, we are interested in those examples where Δ is an elementary operator of length 2 with coefficients from the set $\{J_n(0), J_n(0)^\top, I_n\}$. The reader can easily verify that the kernel of any such elementary operator has the same reflexivity defect as the kernel of one of the following linear transformations on $\mathbb{M}_n$: $\Delta_1(T) = J_n(0)T - TJ_n(0)$, $\Delta_2(T) = J_n(0)TJ_n(0) - T$, $\Delta_3(T) = J_n(0)^\top TJ_n(0) - J_n(0)T$ and $\Delta_4(T) = J_n(0)TJ_n(0)^\top - J_n(0)^\top TJ_n(0)$. For example, consider the kernel of the linear transformation $\tilde{\Delta}(T) = J_n(0)^\top TJ_n(0)^\top - TJ_n(0)$. Using transformations of the form (2) we will show that $\mathrm{rd}(\ker\tilde{\Delta}) = \mathrm{rd}(\ker\Delta_3)$. It suffices to show that $T \in \ker\Delta_3$ if and only if $T^\top P_n \in \ker\tilde{\Delta}$. Indeed, first transpose the equation $J_n(0)^\top TJ_n(0) = J_n(0)T$ and then multiply by P_n on the right side to get $J_n(0)^\top T^\top J_n(0)P_n = T^\top J_n(0)^\top P_n$. Since $P_n^2 = I_n$ and $J_n(0)^\top = P_nJ_n(0)P_n$ we get $J_n(0)^\top(T^\top P_n)J_n(0)^\top = (T^\top P_n)J_n(0)$, i.e., $T^\top P_n \in \ker\tilde{\Delta}$.

Hence, it suffices to determine $\mathrm{rd}(\ker\Delta_i)$ for $i = 1, 2, 3, 4$. First let us introduce some notation. For a given subspace $\mathcal{S} \subset \mathbb{M}_{p,r}$ we define $\mathcal{S} \oplus 0_{i,j} = \{S \oplus 0_{i,j} : S \in \mathcal{S}\} \subseteq \mathbb{M}_{p+i,r+j}$. By Proposition 2.1 one has $\mathrm{rd}(\ker\Delta_1) = \frac{1}{2}n(n-1)$ and Proposition 2.2 yields $\mathrm{rd}(\ker\Delta_2) = 0$. In what follows most of the computation has been done using program Mathematica. If $n = 2l$ for some positive integer l, then the elements of $\ker\Delta_3$ are matrices of the form $\begin{pmatrix} A_1 & \dots & A_{2l}\end{pmatrix}^\top$, where $A_i \in \mathbb{M}_{1,2l}$, $i \in \{1, \dots, 2l\}$, are rows of the form

$$A_i = \begin{cases} \begin{pmatrix} 0 & \dots & 0 & a_1 & \dots & a_{l-j+1}\end{pmatrix} & : i = 2j-1, j \in \{1, \dots, l\}, \\ \begin{pmatrix} 0 & \dots & 0\end{pmatrix} & : i = 2j, j \in \{1, \dots, l\}. \end{cases}$$

It is easy to see that in this case $\ker\Delta_3$ is a space similar to $\mathcal{UT}(l) \oplus 0_{l,l}$. Next, if $n = 2l+1$ for some positive integer l, then the elements of $\ker\Delta_3$ are matrices of the form $\begin{pmatrix} A_1 & \dots & A_{2l+1}\end{pmatrix}^\top$, where $A_i \in \mathbb{M}_{1,2l+1}$, $i \in \{1, \dots, 2l+1\}$, are rows of the form

$$A_i = \begin{cases} \begin{pmatrix} 0 & \dots & 0 & a_1 & \dots & a_{2l-j+2}\end{pmatrix} & : i = 2j-1, j \in \{1, \dots, l+1\}, \\ \begin{pmatrix} 0 & \dots & 0\end{pmatrix} & : i = 2j, j \in \{1, \dots, l\}. \end{cases}$$

One can see that in this case $\ker\Delta_3$ has the same reflexivity defect as the space $\mathcal{A}(l+1, 2l+1)$.

Now consider $\ker\Delta_4$. We will characterize its structure by describing the main diagonal, upper diagonals and lower diagonals of its elements. For a given matrix $T \in \mathbb{M}_n$ let $D(T)$ denote the row vector representing the diagonal of T, let $D_i^U(T)$ denote the row vector representing the ith upper diagonal and let $D_i^L(T)$ denote the row vector representing the ith lower diagonal of T. Let $T \in \ker\Delta_4$. If $n = 2l$ for some positive integer l, then $D(T) = \begin{pmatrix} 0 & \dots & 0\end{pmatrix}$ and for $i = 1, \dots, 2l-1$ we have

$$D_i^U(T) = \begin{cases} \begin{pmatrix} a_j & 0 & a_j & 0 & \dots & a_j\end{pmatrix} & : \text{if } i = 2j-1, j \in \{1, \dots, l\}, \\ \begin{pmatrix} 0 & \dots & 0\end{pmatrix} & : \text{if } i = 2j, j \in \{1, \dots, l-1\}, \end{cases}$$

$$D_i^L(T) = \begin{cases} \begin{pmatrix} a_{l+j} & 0 & a_{l+j} & 0 & \dots & a_{l+j}\end{pmatrix} & : \text{if } i = 2j-1, j \in \{1, \dots, l\}, \\ \begin{pmatrix} 0 & \dots & 0\end{pmatrix} & : \text{if } i = 2j, j \in \{1, \dots, l-1\}. \end{cases}$$

It is easy to see that in this case $\ker\Delta_4$ is similar to $\mathcal{UT}(l) \oplus \mathcal{UT}(l)$. Next, if $n = 2l+1$ for some positive integer l, then $D(T) = \begin{pmatrix} a_1 & 0 & a_1 & 0 & \dots & a_1\end{pmatrix}$ and for $i = 1, \dots, 2l$ we have

$$D_i^U(T) = \begin{cases} \begin{pmatrix} a_{j+1} & 0 & a_{j+1} & 0 & \dots & a_{j+1}\end{pmatrix} & : \text{if } i = 2j, j \in \{1, \dots, l\}, \\ \begin{pmatrix} 0 & \dots & 0\end{pmatrix} & : \text{if } i = 2j-1, j \in \{1, \dots, l\}, \end{cases}$$

$$D_i^L(T) = \begin{cases} \begin{pmatrix} a_{l+j+1} & 0 & a_{l+j+1} & 0 & \dots & a_{l+j+1}\end{pmatrix} & : \text{if } i = 2j, j \in \{1, \dots, l\}, \\ \begin{pmatrix} 0 & \dots & 0\end{pmatrix} & : \text{if } i = 2j-1, j \in \{1, \dots, l\}. \end{cases}$$

It is not hard to see that in this case $\ker\Delta_4$ is similar to $\mathcal{T}(l+1,l+1)\oplus 0_{l,l}$. Hence, Proposition 2.4 yields

$$\mathrm{rd}(\ker\Delta_3)=\begin{cases}\frac{1}{2}l(l-1) & \text{if } n=2l,\ l\in\mathbb{N},\\ \frac{1}{2}l(3l+1) & \text{if } n=2l+1,\ l\in\mathbb{N},\end{cases}$$

$$\mathrm{rd}(\ker\Delta_4)=\begin{cases}l(l-1) & \text{if } n=2l,\ l\in\mathbb{N},\\ l^2 & \text{if } n=2l+1,\ l\in\mathbb{N}.\end{cases}$$

3. Linear matrix pencils

Let $\Delta(T)=A_1TB_1-A_2TB_2$ ($T\in\mathbb{M}_n$), where A_1,A_2 and B_1,B_2 are linearly independent. Note that depending on the invertibility of the coefficients of Δ we can sometimes translate the problem of determining $\mathrm{rd}(\ker\Delta)$ to an already solved one. For example, if A_1 and B_2 are both non-singular, then $\ker\Delta$ is equal to the kernel of the generalized derivation $TB_1B_2^{-1}-A_1^{-1}A_2T$, hence the reflexivity defect can essentially be determined by Proposition 2.1. But since the Jordan structure of the product in general cannot be derived from the Jordan structure of the factors a general formula cannot be obtained in this way. Moreover, parameters that appeared in the formulae for the reflexivity defect in Section 2 were only the corresponding Jordan block sizes. However, one cannot expect the same would happen in general case. To be able to get the explicit general formulae for reflexivity defect of the kernel of Δ we will use the theory of linear matrix pencils and since we will be dealing with pairs of matrices it is reasonable to expect more parameters will appear. Let us introduce some basic notions, for more detailed explanation see [3, Chapter XII].

Let A_1,A_2,B_1,B_2 be n-by-m complex matrices. We say that matrix pencils $A_1+\lambda B_1$ and $A_2+\lambda B_2$ are strictly equivalent if there exist two square non-singular matrices $P\in\mathbb{M}_n$ and $Q\in\mathbb{M}_m$ such that $P(A_1+\lambda B_1)Q=A_2+\lambda B_2$ or, equivalently, $PA_1Q=A_2$ and $PB_1Q=B_2$. For a given matrices $A,B\in\mathbb{M}_{n,m}$ it is said that the linear matrix pencil $A+\lambda B$ is regular if A and B are square matrices and $\det(A+\lambda B)$ is not identically equal to zero. If $n\neq m$ or if $m=n$ and $\det(A+\lambda B)$ is identically zero, then the pencil is singular. Let $A+\lambda B$ be a regular matrix pencil where A and B are n-by-n complex matrices. We can determine the so-called finite and infinite elementary divisors of this pencil in the following way. First we give the pencil $A+\lambda B$ in terms of homogeneous parameters λ and μ, i.e., $\mu A+\lambda B$. Obviously, $\det(\mu A+\lambda B)$ is a homogeneous function of λ and μ. Now determine the greatest common divisor $D_k(\lambda,\mu)$ of all the minors of order k of the matrix pencil $\mu A+\lambda B$ for $k=1,\dots,n$ and put $D_0(\lambda,\mu)=1$. From this the factors polynomials can be obtained in a well-known way

$$i_1(\lambda,\mu)=\frac{D_n(\lambda,\mu)}{D_{n-1}(\lambda,\mu)},\ i_2(\lambda,\mu)=\frac{D_{n-1}(\lambda,\mu)}{D_{n-2}(\lambda,\mu)},\ \dots,\ i_{n-1}=\frac{D_2(\lambda,\mu)}{D_1(\lambda,\mu)},\ i_n(\lambda,\mu)=D_1(\lambda,\mu).$$

Of course, all the $D_k(\lambda,\mu)$ and $i_l(\lambda,\mu)$ are homogeneous polynomials in λ and μ. If we split the invariant polynomials into powers of linear homogeneous polynomials, we obtain the elementary divisors $e_j(\lambda,\mu)$ for $j=1,2,\dots$ of the pencil $\mu A+\lambda B$. Note that if we set $\mu=1$, we get the elementary divisors $e_j(\lambda)$ of the matrix pencil $A+\lambda B$. By the formula $e_j(\lambda,\mu)=\mu^q e_j(\frac{\lambda}{\mu})$ one can obtain from elementary divisor $e_j(\lambda)$ of the degree q the corresponding elementary divisor $e_j(\lambda,\mu)$. In this way all the elementary divisors of the pencil $\mu A+\lambda B$ can be obtained from the elementary divisors of the pencil $A+\lambda B$ with the exception of the elementary divisors of the form μ^q. The elementary divisors of the latter form exist if and only if $\det B=0$ and we call them the infinite elementary divisors of the pencil $A+\lambda B$. The other elementary divisors are called the finite elementary divisors. It is well known that two strictly equivalent regular pencils need to have the same finite and infinite divisors.

Let us introduce the following matrices

$$F_i=\begin{pmatrix}0&1&0&\dots&0\\ \vdots&\ddots&\ddots&\ddots&\vdots\\ \vdots&&\ddots&\ddots&0\\ 0&\dots&\dots&0&1\end{pmatrix}\in\mathbb{M}_{i,i+1},\quad G_i=\begin{pmatrix}1&0&\dots&\dots&0\\ 0&\ddots&\ddots&&\vdots\\ \vdots&\ddots&\ddots&\ddots&\vdots\\ 0&\dots&0&1&0\end{pmatrix}\in\mathbb{M}_{i,i+1}$$

and define $L_i = F_i + \lambda G_i$. It is well known that every matrix pencil $A + \lambda B \in \mathbb{M}_{m,n}$ is strictly equivalent to a matrix pencil of the following block diagonal form

$$0_{w,z} \oplus (L_{a_1} \oplus \cdots \oplus L_{a_e}) \oplus (L^{\mathsf{T}}_{b_1} \oplus \cdots \oplus L^{\mathsf{T}}_{b_f}) \tag{5}$$

$$\oplus ((I_{c_1} + \lambda J_{c_1}(0)) \oplus \cdots \oplus (I_{c_g} + \lambda J_{c_g}(0))) \tag{6}$$

$$\oplus ((\lambda I_{d_1} + J_{d_1}(\alpha_1)) \oplus \cdots \oplus (\lambda I_{d_h} + J_{d_h}(\alpha_h))), \tag{7}$$

called the Kronecker canonical form of pencil $A + \lambda B$, cf. [2,3,7,11]. Here $0_{w,z}$ denotes the w-by-z zero matrix and $a_1 \leqslant \cdots \leqslant a_e$, $b_1 \leqslant \cdots \leqslant b_f$, $c_1 \leqslant \cdots \leqslant c_g$ and $d_1 \leqslant \cdots \leqslant d_h$ are positive integers. The integers a_i, b_j and c_k are uniquely determined by the pair A, B and the part (7) is uniquely determined by A and B up to a permutation of the diagonal blocks. The blocks $0_{w,z}$, L_{a_i} and $L^{\mathsf{T}}_{b_j}$ contain the singularity of the pencil $A + \lambda B$. Here $w = \dim \bigcap_{\lambda \in \mathbb{C}} \operatorname{kel}(A + \lambda B)$ and $z = \dim \bigcap_{\lambda \in \mathbb{C}} \ker(A + \lambda B)$, where $\operatorname{kel}(Y) = \{x \in \mathbb{M}_{1,n} : xY = 0\}$ denotes the left kernel of a matrix $Y \in \mathbb{M}_{n,m}$. The sizes of the blocks L_{a_i} and $L^{\mathsf{T}}_{b_j}$ characterize them completely since there exist polynomial vector columns $x_i(\lambda) = \begin{pmatrix} 1 & -\lambda & \lambda^2 & \ldots & (-1)^{a_i - 1}\lambda^{a_i} \end{pmatrix}^{\mathsf{T}}$ such that $L_{a_i} x_i(\lambda) = 0$ and polynomial vector rows $y_j(\lambda) = \begin{pmatrix} 1 & -\lambda & \lambda^2 & \ldots & (-1)^{b_j - 1}\lambda^{b_j} \end{pmatrix}$ such that $y_j(\lambda) L^{\mathsf{T}}_{b_j} = 0$. That is, the block L_{a_i} has a one-dimensional right kernel spanned by $x_i(\lambda)$ for every λ and the block $L^{\mathsf{T}}_{b_j}$ has a one-dimensional left kernel spanned by $y_j(\lambda)$ for every λ. The part (5) is called the singular part of the Kronecker form of the pencil $A + \lambda B$ and the integers a_i and b_j are called the right (or column) and the left (or row) minimal indices of the pencil $A + \lambda B$, respectively. The regular part of $A + \lambda B$ consists of (6) and (7). Here, (6) contains infinite elementary divisors and is uniquely determined by their degrees. The infinite elementary divisors exist if and only if $\det B = 0$, otherwise there are only finite elementary divisors which are contained in (7) and which also uniquely determine (7). The numbers $-\alpha_j$ are called the eigenvalues of the pencil $A + \lambda B$.

Let Δ be a linear transformation of the form $\Delta(T) = A_1 T B_1 - A_2 T B_2 \quad (T \in \mathbb{M}_n)$, where A_1, A_2 and B_1, B_2 are linearly independent. Let $\epsilon_1, \ldots, \epsilon_p$ and $\eta_1, \ldots, \eta_q$ be the right and the left minimal indices of $A_1 + \lambda A_2$, respectively. Let $k_1, \ldots, k_r$ be the degrees of the infinite elementary divisors and let $(\lambda + \alpha_1)^{l_1}, \ldots, (\lambda + \alpha_s)^{l_s}$ be the finite elementary divisors of $A_1 + \lambda A_2$. Similarly, let $a_1, \ldots, a_e$ and $b_1, \ldots, b_f$ be the right and left minimal indices of $B_1 + \lambda B_2$, respectively. Let $c_1, \ldots, c_g$ be the degrees of the infinite elementary divisors and let $(\lambda + \beta_1)^{d_1}, \ldots, (\lambda + \beta_h)^{d_h}$ be the finite elementary divisors of $B_1 + \lambda B_2$. The Kronecker canonical forms of matrix pencils $A_1 + \lambda A_2$ and $B_1 + \lambda B_2$ yield that coefficients of Δ are similar to the following block diagonal matrices

$$\begin{aligned}
A_1 &\sim 0_{u,v} \oplus (F_{\epsilon_1} \oplus \cdots \oplus F_{\epsilon_p}) \oplus (F^{\mathsf{T}}_{\eta_1} \oplus \cdots \oplus F^{\mathsf{T}}_{\eta_q}) \oplus (I_{k_1} \oplus \cdots \oplus I_{k_r}) \\
&\qquad \oplus (J_{l_1}(\alpha_1) \oplus \cdots \oplus J_{l_s}(\alpha_s)), \\
A_2 &\sim 0_{u,v} \oplus (G_{\epsilon_1} \oplus \cdots \oplus G_{\epsilon_p}) \oplus (G^{\mathsf{T}}_{\eta_1} \oplus \cdots \oplus G^{\mathsf{T}}_{\eta_q}) \oplus (J_{k_1}(0) \oplus \cdots \oplus J_{k_r}(0)) \\
&\qquad \oplus (I_{l_1} \oplus \cdots \oplus I_{l_s}), \\
B_1 &\sim 0_{w,z} \oplus (F_{a_1} \oplus \cdots \oplus F_{a_e}) \oplus (F^{\mathsf{T}}_{b_1} \oplus \cdots \oplus F^{\mathsf{T}}_{b_f}) \oplus (I_{c_1} \oplus \cdots \oplus I_{c_g}) \\
&\qquad \oplus (J_{d_1}(\beta_1) \oplus \cdots \oplus J_{d_h}(\beta_h)), \\
B_2 &\sim 0_{w,z} \oplus (G_{a_1} \oplus \cdots \oplus G_{a_e}) \oplus (G^{\mathsf{T}}_{b_1} \oplus \cdots \oplus G^{\mathsf{T}}_{b_f}) \oplus (J_{c_1}(0) \oplus \cdots \oplus J_{c_g}(0)) \\
&\qquad \oplus (I_{d_1} \oplus \cdots \oplus I_{d_h}).
\end{aligned} \tag{8}$$

According to the notation in (8) define the following non-negative integers

$$R_{\epsilon,b}(i,j) = \epsilon_i b_j,$$

$$R_{k,d}(i,j) = \frac{\delta_{\beta_j,0}}{2} \min\{k_i,\ d_j\}(\min\{k_i,\ d_j\} - 1),$$

$$R_{l,c}(i,j) = \frac{\delta_{\alpha_i,0}}{2}\min\{l_i,\ c_j\}(\min\{l_i,\ c_j\}-1),$$
$$R_{l,d}(i,j) = \frac{\delta_{\alpha_i\beta_j,1}}{2}\min\{l_i,\ d_j\}(\min\{l_i,\ d_j\}-1),$$
$$R_{\epsilon,c}(i,j) = \begin{cases} \frac{1}{2}c_j(c_j-1) & \epsilon_i \geqslant c_j, \\ \epsilon_i(c_j-\frac{1}{2}(\epsilon_i+1)) & \epsilon_i < c_j, \end{cases}$$
$$R_{\epsilon,d}(i,j) = \begin{cases} \frac{1}{2}d_j(d_j-1) & \epsilon_i \geqslant d_j, \\ \epsilon_i(d_j-\frac{1}{2}(\epsilon_i+1)) & \epsilon_i < d_j, \end{cases}$$
$$R_{k,b}(i,j) = \begin{cases} \frac{1}{2}k_i(k_i-1) & b_j \geqslant k_i, \\ b_j(k_i-\frac{1}{2}(b_j+1)) & b_j < k_i, \end{cases}$$
$$R_{l,b}(i,j) = \begin{cases} \frac{1}{2}l_i(l_i-1) & b_j \geqslant l_i, \\ b_j(l_i-\frac{1}{2}(b_j+1)) & b_j < l_i. \end{cases}$$

The following theorem provides a general formula for the reflexivity defect of an elementary operator of length 2 and is the main result of this paper.

Theorem 3.1. *Let $A_i,\ B_i \in \mathbb{M}_n$ and let their Kronecker canonical decompositions be as in (8). Define $\Delta(T) = A_1TB_1 - A_2TB_2$, $T \in \mathbb{M}_n$. Then*

$$\begin{aligned}\mathrm{rd}(\ker\Delta) = &\sum_{i=1}^{p}\sum_{j=1}^{f} R_{\epsilon,b}(i,j) + \sum_{i=1}^{p}\sum_{j=1}^{g} R_{\epsilon,c}(i,j) + \sum_{i=1}^{p}\sum_{j=1}^{h} R_{\epsilon,d}(i,j) + \sum_{i=1}^{r}\sum_{j=1}^{f} R_{k,b}(i,j) \\ &+ \sum_{i=1}^{r}\sum_{j=1}^{h} R_{k,d}(i,j) + \sum_{i=1}^{s}\sum_{j=1}^{f} R_{l,b}(i,j) + \sum_{i=1}^{s}\sum_{j=1}^{g} R_{l,c}(i,j) + \sum_{i=1}^{s}\sum_{j=1}^{h} R_{l,d}(i,j).\end{aligned}$$

Proof. Define the following elementary operators

$$\begin{aligned}
&\Delta_{ij}^{1,S}(T) = F_{\epsilon_i}TF_{a_j} - G_{\epsilon_i}TG_{a_j},\ \Delta_{ij}^{2,S}(T) = F_{\epsilon_i}TF_{b_j}^{\mathsf{T}} - G_{\epsilon_i}TG_{b_j}^{\mathsf{T}},\\
&\Delta_{ij}^{3,S}(T) = F_{\epsilon_i}T - G_{\epsilon_i}TJ_{c_j}(0),\ \Delta_{ij}^{4,S}(T) = F_{\epsilon_i}TJ_{d_j}(\beta_j) - G_{\epsilon_i}T,\\
&\Delta_{ij}^{5,S}(T) = F_{\eta_i}^{\mathsf{T}}TF_{a_j} - G_{\eta_i}^{\mathsf{T}}TG_{a_j},\ \Delta_{ij}^{6,S}(T) = F_{\eta_i}^{\mathsf{T}}TF_{b_j}^{\mathsf{T}} - G_{\eta_i}^{\mathsf{T}}TG_{b_j}^{\mathsf{T}},\\
&\Delta_{ij}^{7,S}(T) = F_{\eta_i}^{\mathsf{T}}T - G_{\eta_i}^{\mathsf{T}}TJ_{c_j}(0),\ \Delta_{ij}^{8,S}(T) = F_{\eta_i}^{\mathsf{T}}TJ_{d_j}(\beta_j) - G_{\eta_i}^{\mathsf{T}}T,\\
&\Delta_{ij}^{9,S}(T) = TF_{a_j} - J_{k_i}(0)TG_{a_j},\ \Delta_{ij}^{10,S}(T) = TF_{b_j}^{\mathsf{T}} - J_{k_i}(0)TG_{b_j}^{\mathsf{T}},\\
&\Delta_{ij}^{11,R}(T) = T - J_{k_i}(0)TJ_{c_j}(0),\ \Delta_{ij}^{12,R}(T) = TJ_{d_j}(\beta_j) - J_{k_i}(0)T,\\
&\Delta_{ij}^{13,S}(T) = J_{l_i}(\alpha_i)TF_{a_j} - TG_{a_j},\ \Delta_{ij}^{14,S}(T) = J_{l_i}(\alpha_i)TF_{b_j}^{\mathsf{T}} - TG_{b_j}^{\mathsf{T}},\\
&\Delta_{ij}^{15,R}(T) = J_{l_i}(\alpha_i)T - TJ_{c_j}(0),\ \Delta_{ij}^{16,R}(T) = J_{l_i}(\alpha_i)TJ_{d_j}(\beta_j) - T.
\end{aligned}$$

The letter R in the superscript denotes that the given elementary operator has coefficients that appear only in the regular parts of pencils $A_1 + \lambda A_2$ and $B_1 + \lambda B_2$ and, similarly, the letter S denotes that coefficients are of mixed type, some from regular and some from singular parts. The letters in the subscript are the non-negative integers from (8) with appropriate choice of the domain of i and j. Using the block diagonal structure of the coefficients of Δ one can get the following block decomposition of ker Δ

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \left[\ker \Delta_{ij}^{1,S}\right]_{i,j=1}^{p,e} & \left[\ker \Delta_{ij}^{2,S}\right]_{i,j=1}^{p,f} & \left[\ker \Delta_{ij}^{3,S}\right]_{i,j=1}^{p,g} & \left[\ker \Delta_{ij}^{4,S}\right]_{i,j=1}^{p,h} \\ 0 & \left[\ker \Delta_{ij}^{5,S}\right]_{i,j=1}^{q,e} & \left[\ker \Delta_{ij}^{6,S}\right]_{i,j=1}^{q,f} & \left[\ker \Delta_{ij}^{7,S}\right]_{i,j=1}^{q,g} & \left[\ker \Delta_{ij}^{8,S}\right]_{i,j=1}^{q,h} \\ 0 & \left[\ker \Delta_{ij}^{9,S}\right]_{i,j=1}^{r,e} & \left[\ker \Delta_{ij}^{10,S}\right]_{i,j=1}^{r,f} & \left[\ker \Delta_{ij}^{11,R}\right]_{i,j=1}^{r,g} & \left[\ker \Delta_{ij}^{12,R}\right]_{i,j=1}^{r,h} \\ 0 & \left[\ker \Delta_{ij}^{13,S}\right]_{i,j=1}^{s,e} & \left[\ker \Delta_{ij}^{14,S}\right]_{i,j=1}^{s,f} & \left[\ker \Delta_{ij}^{15,R}\right]_{i,j=1}^{s,g} & \left[\ker \Delta_{ij}^{16,R}\right]_{i,j=1}^{s,h} \end{pmatrix}. \tag{9}$$

Proposition 2.2 yields $\mathrm{rd}(\ker \Delta_{ij}^{16,R}) = R_{l,d}(i,j)$ and that $\ker \Delta_{ij}^{11,R} = \{0\}$, therefore we get $\mathrm{rd}(\ker \Delta_{ij}^{11,R}) = 0$. By Proposition 2.1 one has $\mathrm{rd}(\ker \Delta_{ij}^{12,R}) = R_{k,d}(i,j)$ and $\mathrm{rd}(\ker \Delta_{ij}^{15,R}) = R_{l,c}(i,j)$. In what follows most of the computation has been done using program Mathematica. Next, if we can express $\mathrm{rd}(\ker \Delta_{ij}^{1,S})$, then we can also express $\mathrm{rd}(\ker \Delta_{ij}^{6,S})$. Indeed, let $T \in \ker \Delta_{ij}^{1,S}$, hence $F_{\epsilon_i} T F_{a_j} = G_{\epsilon_i} T G_{a_j}$. Transposing this equation we get $F_{a_j}^{\mathsf{T}} T^{\mathsf{T}} F_{\epsilon_i}^{\mathsf{T}} = G_{a_j}^{\mathsf{T}} T^{\mathsf{T}} G_{\epsilon_i}^{\mathsf{T}}$, hence T^{T} is the element of the kernel of the transformation $S \mapsto F_{a_j}^{\mathsf{T}} S F_{\epsilon_i}^{\mathsf{T}} - G_{a_j}^{\mathsf{T}} S G_{\epsilon_i}^{\mathsf{T}}$. If $\mathrm{rd}(\ker \Delta_{ij}^{1,S}) = f_1(\epsilon_i, a_j)$ for some function of two variables f_1, then $\mathrm{rd}(\ker \Delta_{ij}^{6,S}) = f_1(b_j, \eta_i)$ since transposing preserves the reflexivity defect. It is easy to see that $\ker \Delta_{ij}^{1,S} = \{0\}$ if $\epsilon_i \geqslant a_j$. Otherwise, if $\epsilon_i < a_j$, then

$$\ker \Delta_{ij}^{1,S} = \left\{ \begin{pmatrix} 0 & \ldots & 0 & t_1 & \ldots & \ldots & t_{a_j-\epsilon_i} \\ \vdots & & \iddots & \iddots & & \iddots & 0 \\ 0 & \iddots & \iddots & & \iddots & \iddots & \vdots \\ t_1 & \ldots & \ldots & t_{a_j-\epsilon_i} & 0 & \ldots & 0 \end{pmatrix} : t_l \in \mathbb{C} \right\}.$$

Using transformations of the form (2) which preserve the reflexivity defect we get

$$\mathrm{rd}(\ker \Delta_{ij}^{1,S}) = \begin{cases} \mathrm{rd}(\mathcal{B}(\epsilon_i + 1, a_j)) & : \epsilon_i < a_j, \\ 0 & : \epsilon_i \geqslant a_j. \end{cases}$$

Now Proposition 2.4 yields $\mathrm{rd}(\ker \Delta_{ij}^{1,S}) = 0$ and by the discussion above we also have $\mathrm{rd}(\ker \Delta_{ij}^{6,S}) = 0$. As for $\Delta_{ij}^{2,S}$, one can see that $\ker \Delta_{ij}^{2,S} = \mathcal{T}(\epsilon_i + 1, b_j + 1)$ and by Proposition 2.4 we have $\mathrm{rd}(\ker \Delta_{ij}^{2,S}) = R_{\epsilon,b}(i,j)$. Next, if we can express $\mathrm{rd}(\ker \Delta_{ij}^{3,S})$, then we can also express $\mathrm{rd}(\ker \Delta_{ij}^{10,S})$. Indeed, let $T \in \ker \Delta_{ij}^{3,S}$, hence $F_{\epsilon_i} T = G_{\epsilon_i} T J_{c_j}(0)$. Transposing this equation we get $T^{\mathsf{T}} F_{\epsilon_i}^{\mathsf{T}} = J_{c_j}(0)^{\mathsf{T}} T^{\mathsf{T}} G_{\epsilon_i}^{\mathsf{T}}$. Now multiply the latter equation by P_{c_j} on the left and consider the facts $P_{c_j}^2 = I_{c_j}$ and $P_{c_j} J_{c_j}(0)^{\mathsf{T}} P_{c_j} = J_{c_j}(0)$ to get $(P_{c_j} T^{\mathsf{T}}) F_{\epsilon_i}^{\mathsf{T}} = J_{c_j}(0)(P_{c_j} T^{\mathsf{T}}) G_{\epsilon_i}^{\mathsf{T}}$. Hence, $P_{c_j} T^{\mathsf{T}}$ is the element of the kernel of the transformation $S \mapsto S F_{\epsilon_i}^{\mathsf{T}} - J_{c_j}(0) S G_{\epsilon_i}^{\mathsf{T}}$ and therefore if $\mathrm{rd}(\ker \Delta_{ij}^{3,S}) = f_2(\epsilon_i, c_j)$ for some function of two variables f_2, then $\mathrm{rd}(\ker \Delta_{ij}^{10,S}) = f_2(b_j, k_i)$ since transposing and multiplying by invertible matrix preserves the reflexivity defect. It is not hard to see that

$$\ker \Delta_{ij}^{3,S} = \begin{cases} \begin{pmatrix} \mathcal{UT}(c_j) \\ 0_{\epsilon_i - c_j + 1, c_j} \end{pmatrix} & : \epsilon_i \geqslant c_j, \\ \mathcal{A}(\epsilon_i + 1, c_j) & : \epsilon_i < c_j. \end{cases}$$

By Proposition 2.4 we get $\mathrm{rd}(\ker \Delta_{ij}^{3,S}) = R_{\epsilon,c}(i,j)$ and by the discussion above we have $\mathrm{rd}(\ker \Delta_{ij}^{10,S}) = R_{k,b}(i,j)$. Similarly, if we can express $\mathrm{rd}(\ker \Delta_{ij}^{4,S})$, then we can also express $\mathrm{rd}(\ker \Delta_{ij}^{14,S})$. Indeed, let $T \in \ker \Delta_{ij}^{4,S}$, hence $F_{\epsilon_i} T J_{d_j}(\beta_j) = G_{\epsilon_i} T$. Transposing this equation we get $J_{d_j}(\beta_j)^{\mathsf{T}} T^{\mathsf{T}} F_{\epsilon_i}^{\mathsf{T}} = T^{\mathsf{T}} G_{\epsilon_i}^{\mathsf{T}}$. Now multiply the latter equation by P_{d_j} on the left and consider the facts $P_{d_j}^2 = I_{d_j}$ and $P_{d_j} J_{d_j}(\beta_j)^{\mathsf{T}} P_{d_j} = J_{d_j}(\beta_j)$

to get $J_{d_j}(\beta_j)(P_{d_j}T^\top)F_{\epsilon_i}^\top = (P_{d_j}T^\top)G_{\epsilon_i}^\top$. Hence, $P_{d_j}T^\top$ is the element of the kernel of the transformation $S \mapsto J_{d_j}(\beta_j)SF_{\epsilon_i}^\top - SG_{\epsilon_i}^\top$ and therefore if $\mathrm{rd}(\ker \Delta_{ij}^{4,S}) = f_3(\epsilon_i, d_j, \beta_j)$ for some function of three variables f_3, then $\mathrm{rd}(\ker \Delta_{ij}^{14,S}) = f_3(b_j, l_i, \alpha_i)$ since transposing and multiplying by invertible matrix preserves the reflexivity defect. Let $T \in \ker \Delta_{ij}^{4,S}$. We can describe T coefficient-wise in the following way. The coefficients in the last row, i.e., $t_{\epsilon_i+1,1}, \ldots, t_{\epsilon_i+1,d_j}$ are parameters. The coefficients in the first column are of the form $t_{k1} = \beta_j^{\epsilon_i-k+1} t_{\epsilon_i+1,1}$ for $k = 1, \ldots, \epsilon_i+1$. The rest of the coefficients can be expressed as $t_{kl} = t_{k+1,l-1} + \beta_j t_{k+1,l}$ for $k = 1, \ldots, \epsilon_i$ and $l = 2, \ldots, d_j$. First we consider the case when $\beta_j = 0$. If $\epsilon_i \geqslant d_j$, then we get

$$\ker \Delta_{ij}^{4,S} = \left\{ \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & t_1 \\ \vdots & \iddots & \iddots & \vdots \\ 0 & \iddots & \iddots & \vdots \\ t_1 & \cdots & \cdots & t_{d_j} \end{pmatrix} : t_l \in \mathbb{C} \right\} \tag{10}$$

and if $\epsilon_i < d_j$, then

$$\ker \Delta_{ij}^{4,S} = \left\{ \begin{pmatrix} 0 & \cdots & 0 & t_1 & \cdots & t_{d_j-\epsilon_i} \\ \vdots & \iddots & \iddots & & & \vdots \\ 0 & \iddots & & & & \vdots \\ t_1 & \cdots & \cdots & \cdots & \cdots & t_{d_j} \end{pmatrix} : t_l \in \mathbb{C} \right\}. \tag{11}$$

If $\beta_j \neq 0$, then we can use elementary row operations to transform $\ker \Delta_{ij}^{4,S}$ to the form (10) if $\epsilon_i \geqslant d_j$ and to the form (11) if $\epsilon_i < d_j$. Hence, one can easily see that

$$\ker \Delta_{ij}^{4,S} \sim \begin{cases} \begin{pmatrix} 0_{\epsilon_i-d_j+1,d_j} \\ \mathcal{UT}(d_j) \end{pmatrix} & : \epsilon_i \geqslant d_j, \\ \mathcal{A}(\epsilon_i+1, d_j) & : \epsilon_i < d_j. \end{cases}$$

Since transformations used are of the form (2) and therefore preserve the reflexivity defect, one has $\mathrm{rd}(\ker \Delta_{ij}^{4,S}) = R_{\epsilon,d}(i,j)$. By the discussion above we also get $\mathrm{rd}(\ker \Delta_{ij}^{14,S}) = R_{l,b}(i,j)$. One can easily see that $\ker \Delta_{ij}^{5,S} = \{0\}$, therefore $\mathrm{rd}(\ker \Delta_{ij}^{5,S}) = 0$. Next, if we can express $\mathrm{rd}(\ker \Delta_{ij}^{9,S})$, then we can also express $\mathrm{rd}(\ker \Delta_{ij}^{7,S})$. Indeed, let $T \in \ker \Delta_{ij}^{9,S}$, hence $TF_{a_j} = J_{k_i}(0)TG_{a_j}$. Transposing this equation we get $F_{a_j}^\top T^\top = G_{a_j}^\top T^\top J_{k_i}(0)^\top$. Now multiply the latter equation by P_{k_i} on the right and consider the facts $P_{k_i}^2 = I_{k_i}$ and $P_{k_i}J_{k_i}(0)^\top P_{k_i} = J_{k_i}(0)$ to get $F_{a_j}^\top(T^\top P_{k_i}) = G_{a_j}^\top(T^\top P_{k_i})J_{k_i}(0)$. Hence, $T^\top P_{k_i}$ is the element of the kernel of the transformation $S \mapsto F_{a_j}^\top S - G_{a_j}^\top SJ_{k_i}(0)$ and therefore if $\mathrm{rd}(\ker \Delta_{ij}^{9,S}) = f_4(k_i, a_j)$ for some function of two variables f_4, then $\mathrm{rd}(\ker \Delta_{ij}^{7,S}) = f_4(c_j, \eta_i)$ since transposing and multiplying by invertible matrix preserves the reflexivity defect. It is not hard to see that $\ker \Delta_{ij}^{9,S} = \{0\}$, hence $\mathrm{rd}(\ker \Delta_{ij}^{9,S}) = \mathrm{rd}(\ker \Delta_{ij}^{7,S}) = 0$. Next, if we can express $\mathrm{rd}(\ker \Delta_{ij}^{13,S})$, then we can also express $\mathrm{rd}(\ker \Delta_{ij}^{8,S})$. Indeed, let $T \in \ker \Delta_{ij}^{13,S}$, hence $J_{l_i}(\alpha_i)TF_{a_j} = TG_{a_j}$. Transposing this equation we get $F_{a_j}^\top T^\top J_{l_i}(\alpha_i)^\top = G_{a_j}^\top T^\top$. Now multiply the latter equation by P_{l_i} on the right and consider the facts $P_{l_i}^2 = I_{l_i}$

and $P_{l_i}J_{l_i}(\alpha_i)^\top P_{l_i} = J_{l_i}(\alpha_i)$ to get $F_{a_j}^\top(T^\top P_{l_i})J_{l_i}(\alpha_i) = G_{a_j}^\top(T^\top P_{l_i})$. Hence, $T^\top P_{l_i}$ is the element of the kernel of the transformation $S \mapsto F_{a_j}^\top S J_{l_i}(\alpha_i) - G_{a_j}^\top S$ and therefore if $\mathrm{rd}(\ker \Delta_{ij}^{13,S}) = f_5(l_i, a_j, \alpha_i)$ for some function of three variables f_5, then $\mathrm{rd}(\ker \Delta_{ij}^{8,S}) = f_5(d_j, \eta_i, \beta_j)$ since transposing and multiplying by invertible matrix preserves the reflexivity defect. It is not hard to see that $\ker \Delta_{ij}^{13,S} = \{0\}$, hence $\mathrm{rd}(\ker \Delta_{ij}^{13,S}) = \mathrm{rd}(\ker \Delta_{ij}^{8,S}) = 0$. Since $\ker \Delta$ has decomposition of the form (9) we can use the formula in (3) which completes the proof of the theorem. $\square$

Let $A_i,\ B_i \in \mathbb{M}_n$ and define $\Delta(T) = A_1TB_1 - A_2TB_2, T \in \mathbb{M}_n$. The formula for $\mathrm{rd}(\ker \Delta)$ in Theorem 3.1 simplifies if we have additional assumptions on coefficients of Δ. For example, if $A_1 + \lambda A_2$ is regular and $B_1 + \lambda B_2$ is singular matrix pencil, then Theorem 3.1 yields

$$\begin{aligned}\mathrm{rd}(\ker \Delta) &= \sum_{i=1}^{r}\sum_{j=1}^{f} R_{k,b}(i,j) + \sum_{i=1}^{r}\sum_{j=1}^{h} R_{k,d}(i,j) + \sum_{i=1}^{s}\sum_{j=1}^{f} R_{l,b}(i,j)\\ &\quad + \sum_{i=1}^{s}\sum_{j=1}^{g} R_{l,c}(i,j) + \sum_{i=1}^{s}\sum_{j=1}^{h} R_{l,d}(i,j).\end{aligned}$$

Next, if $A_1 + \lambda A_2$ is singular and $B_1 + \lambda B_2$ is regular matrix pencil, then

$$\begin{aligned}\mathrm{rd}(\ker \Delta) &= \sum_{i=1}^{p}\sum_{j=1}^{g} R_{\epsilon,c}(i,j) + \sum_{i=1}^{p}\sum_{j=1}^{h} R_{\epsilon,d}(i,j) + \sum_{i=1}^{r}\sum_{j=1}^{h} R_{k,d}(i,j)\\ &\quad + \sum_{i=1}^{s}\sum_{j=1}^{g} R_{l,c}(i,j) + \sum_{i=1}^{s}\sum_{j=1}^{h} R_{l,d}(i,j).\end{aligned}$$

Moreover, if $A_1 + \lambda A_2$ and $B_1 + \lambda B_2$ are both regular matrix pencils, then

$$\mathrm{rd}(\ker \Delta) = \sum_{i=1}^{r}\sum_{j=1}^{h} R_{k,d}(i,j) + \sum_{i=1}^{s}\sum_{j=1}^{g} R_{l,c}(i,j) + \sum_{i=1}^{s}\sum_{j=1}^{h} R_{l,d}(i,j).$$

Note that Proposition 2.1 and Proposition 2.2 are actually special cases of Theorem 3.1 since determining the Jordan form of an arbitrary matrix $A \in M_n$ is equivalent to finding the (finite) elementary divisors of the regular pencil $\lambda I + A$, see [3].

4. Application of the Theorem 3.1

Let us remember Example 2.5. We were computing reflexivity defect of the kernel of the operators $\Delta_1(T) = J_n(0)T - TJ_n(0),\ \Delta_2(T) = J_n(0)TJ_n(0) - T,\ \Delta_3(T) = J_n(0)^\top TJ_n(0) - J_n(0)T$ and $\Delta_4(T) = J_n(0)TJ_n(0)^\top - J_n(0)^\top TJ_n(0)$. For this purpose we actually had to compute the kernels of operators Δ_i $(i = 1, 2, 3, 4)$ and their reflexivity defects. Thus, we want to apply Theorem 3.1 that enables us to avoid this kind of computation. Instead, we need to determine the Kronecker structure of the appropriate matrix pencils.

Let A and B be n-by-n complex matrices. In [11] it is explained that in order to get information about the Kronecker structure of a matrix pencil $A + \lambda B$ one needs not to actually compute its Kronecker canonical form. From a numerical point of view it is more reasonable to compute the so-called generalized Schur form of the pencil $A + \lambda B$, i.e., a quasitriangular form obtained under unitary transformations,

$$\begin{pmatrix} A_\epsilon + \lambda B_\epsilon & * & * & * \\ 0 & A_\infty + \lambda B_\infty & * & * \\ 0 & 0 & A_f + \lambda B_f & * \\ 0 & 0 & 0 & A_\eta + \lambda B_\eta \end{pmatrix}.$$

Here $A_\epsilon + \lambda B_\epsilon$ is a singular matrix pencil containing the right (column) Kronecker structure of pencil $A + \lambda B$, $A_\infty + \lambda B_\infty$ is a square regular pencil containing the infinite elementary divisors of $A + \lambda B$, $A_f + \lambda B_f$ is a square regular pencil containing the finite elementary divisors of $A + \lambda B$ and $A_\eta + \lambda B_\eta$ is a singular matrix pencil containing the left (row) Kronecker structure of $A + \lambda B$. Each of these diagonal blocks has finer, the so-called staircase structure, which altogether completely determine the Kronecker canonical form of the matrix pencil $A + \lambda B$. There exist algorithms implemented in the commercial software that enable us to compute the generalized Schur form of a matrix pencil and hence enable us to determine its Kronecker canonical structure. In this way we will be able to obtain the result of Example 2.5 in much simpler way, using Theorem 3.1. All the calculations of the Kronecker structure of given matrix pencils in the following example were carried out using command `kroneck` in the program SCILAB.

Example 4.1. Let Δ be a linear transformation of the form $\Delta(T) = A_1 T B_1 - A_2 T B_2,\ (T \in \mathbb{M}_n)$, where $A_1,\ A_2,\ B_1,\ B_2 \in \{J_n(0),\ J_n(0)^\top,\ I_n\}$. In Example 2.5 we already established that the kernel of any such elementary operator has the same reflexivity defect as the kernel of one of the following linear transformations on $\mathbb{M}_n$: $\Delta_1(T) = J_n(0)T - TJ_n(0)$, $\Delta_2(T) = J_n(0)TJ_n(0) - T$, $\Delta_3(T) = J_n(0)^\top TJ_n(0) - J_n(0)T$ and $\Delta_4(T) = J_n(0)TJ_n(0)^\top - J_n(0)^\top TJ_n(0)$. As we already know, Proposition 2.1 yields $\mathrm{rd}(\ker \Delta_1) = \frac{1}{2}n(n-1)$ and by Proposition 2.2 one has $\mathrm{rd}(\ker \Delta_2) = 0$. Using Theorem 3.1 we will determine $\mathrm{rd}(\ker \Delta_3)$ and $\mathrm{rd}(\ker \Delta_4)$. We will use the notation of (8). First consider $\mathrm{rd}(\ker \Delta_3)$. We need to determine the Kronecker structure of the matrix pencils $A_1 + \lambda A_2 = J_n(0)^\top + \lambda J_n(0)$ and $B_1 + \lambda B_2 = J_n(0) + \lambda I_n$. Obviously the latter is a regular pencil which is already in the generalized Schur form and has only finite elementary divisors. Using command `kroneck` we get $h = 1$, $d_1 = n$ and $\beta_1 = 0$. The other indices do not appear. Next, the regularity of the pencil $J_n(0)^\top + \lambda J_n(0)$ depends on n. If n is even, then $J_n(0)^\top + \lambda J_n(0)$ is a regular pencil that has finite and infinite elementary divisors. Using `kroneck` we get $r = 1$, $k_1 = \frac{n}{2}$ and $s = 1$, $l_1 = \frac{n}{2}$, $\alpha_1 = 0$. The other indices do not appear. If n is odd, then $J_n(0)^\top + \lambda J_n(0)$ is a singular pencil that has no regular structure. We get $p = 1$, $\epsilon_1 = \frac{n-1}{2}$ and $q = 1$, $\eta_1 = \frac{n-1}{2}$. Again, the other indices do not appear. Hence, if $n = 2l$ for some $l \in \mathbb{N}$, we get $\mathrm{rd}(\ker \Delta_3) = R_{k,d}(1,1) = \frac{1}{2}l(l-1)$ and if $n = 2l+1$ for some $l \in \mathbb{N}$, we have $\mathrm{rd}(\ker \Delta_3) = R_{\epsilon,d}(1,1) = \frac{1}{2}l(3l+1)$.

Now consider $\mathrm{rd}(\ker \Delta_4)$. Here we need to determine the Kronecker structure of the matrix pencils $A_1 + \lambda A_2 = J_n(0) + \lambda J_n(0)^\top$ and $B_1 + \lambda B_2 = J_n(0)^\top + \lambda J_n(0)$. Similarly as before we get that if n is even, then $r = 1$, $k_1 = \frac{n}{2}$; $s = 1$, $l_1 = \frac{n}{2}$, $\alpha_1 = 0$ and $g = 1$, $c_1 = \frac{n}{2}$; $h = 1$, $d_1 = \frac{n}{2}$, $\beta_1 = 0$. The other indices do not appear. If n is odd, then $p = 1$, $\epsilon_1 = \frac{n-1}{2}$; $q = 1$, $\eta_1 = \frac{n-1}{2}$ and $e = 1$, $a_1 = \frac{n-1}{2}$; $f = 1$, $b_1 = \frac{n-1}{2}$. Again, the other indices do not appear. Hence, if $n = 2l$ for some $l \in \mathbb{N}$, we get $\mathrm{rd}(\ker \Delta_4) = R_{k,d}(1,1) + R_{l,c}(1,1) = l(l-1)$ and if $n = 2l+1$ for some $l \in \mathbb{N}$, we have $\mathrm{rd}(\ker \Delta_4) = R_{\epsilon,b}(1,1) = l^2$.

Remark 4.2. All the results of this paper can be easily extended to the case of the complex rectangular matrices. However, a similar approach for elementary operators of length k for $k \geqslant 3$ would require canonical forms for k-tuples of matrices. In terms of matrix pencils one needs to look at the matrix pencils of k variables.

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